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High-field series expansions and critical properties for the three-state Potts model

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Abstract. High-field series expansions are presented for the three-state Potts model on the square, simple cubic and body centred cubic lattices. Estimates of critical exponents for two dimensions are $\beta = 0{\cdot}1064 \pm 0{\cdot}0005$, $\gamma' = 1{\cdot}50 \pm 0{\cdot}04$ and $\delta = 15{\cdot}5 \pm 1{\cdot}5$. It is plausible that the three-dimensional model undergoes a second-order phase transition with exponents $\beta = 0{\cdot}203 \pm 0{\cdot}004$, $\gamma' = 1{\cdot}18 \pm 0{\cdot}05$ and $\delta = 7{\cdot}0 \pm 0{\cdot}3$.

1. Introduction

The three-state Potts model can be characterised by the Hamiltonian

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \boldsymbol{s}_i \,.\, \boldsymbol{s}_j - H \sum_{i=1}^{N} s_i^x - H_\perp \sum_{i=1}^{N} s_i^y \tag{1}$$

where s_i is a unit vector which is permitted to point in one of only three directions making angles of 0, $2\pi/3$ and $4\pi/3$ radians respectively with the x axis. Other symbols have their usual meaning. The three-state Potts model was introduced by Domb (Domb 1974) and should perhaps be called the 'Domb model'. It was generalised to an arbitrary number of states per site by Potts (1952).

Like the Ising and XY models the three-state Potts model seemed for many years to be of primarily theoretical interest. Recently Mukamel *et al* (1976) have shown that cubic ferromagnets with three easy axes subject to a moderate magnetic field along the [111] direction should behave like the three-state Potts model (see also Kim *et al* (1975)). Barbara *et al* (1978) have made an experimental study of one such substance, Dy Al_2. The three-state Potts model should also be realised in adsorbed monolayers of rare gas molecules on substances such as graphite (Alexander 1975, Berker *et al* 1978).

Figure 1 is a schematic illustration of the phase diagram of the model as previously discussed by Straley and Fisher (1973) and others. Three wings parallel to the temperature axis each represent a coexistence surface between two of the ordered phases; these wings meet in a line of triple points OQ. Each of three webs represents a coexistence surface between one of the ordered phases and the disordered or paramagnetic phase; two webs and a wing also meet in a line of triple points, t_iQ. The free edges of the wing and of the web coexistence surfaces are lines of critical points. The points t_i are tricritical points and Q is a quadruple point.

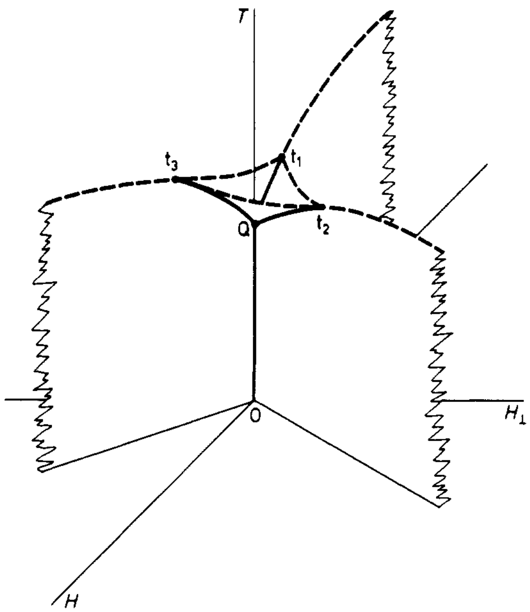


Figure 1. Schematic phase diagram of the three-state Potts model. Heavy full curves are lines of triple points where three coexistence surfaces meet. Coexistence surfaces terminate in lines of critical points indicated by broken curves in the diagram. Q is a quadruple point and t_i are triple points.

A fundamental and controversial question concerning the three-state Potts model in three dimensions is whether the zero-field transition is first or second order. In other words do finite webs exist or do the three tricritical points t_i coalesce with Q, which then becomes a special symmetrical tri-critical point? Mean field theory or Landau theory (Straley and Fisher 1973) predict a finite web and thus, at $H = 0$, a first-order transition at Q independent of the dimension d of the lattice. On the other hand Baxter (1973) has proved that for $d = 2$ the transition is continuous and thus Q is a tricritical point.

Approximate renormalisation-group calculations by Golner (1973) for $d = 3$ and in for ϵ-expansion by Amit and Shcherbakov (1974) predict a first-order transition. Levy and Sudano (1978) also predict a first-order transition for $d = 3$ using the cluster variation method. The question is difficult to settle experimentally but Barbara *et al* (1978) report measurements favouring a first-order transition. On the other hand Burkhardt *et al* (1976) find a second-order transition via the Kadanoff variational method.

Analysis of exact series expansions can yield accurate estimates of critical temperatures, exponents and amplitudes *provided the phase transition is second order.* However, the method of series expansions is not well suited to the problem of ascertaining the form of the phase transition. Using high-temperature series expansions Ditzian and Oitmaa (1975) concluded that the phase transition of the three-state Potts model for $d = 3$ is first order. In contrast Straley and Fisher (1973), Straley (1974) and Enting (1974a), relying mainly on low-temperature series expansions, favour a second-order transition and report estimates of the critical exponents.

In this paper we study the three-state Potts model in two and in three dimensions through both high-field and low-temperature expansions. In § 2 we discuss the method of derivation of the high-field expansion of the free energy of the three-state Potts model on bipartite lattices. For the BCC lattice we have obtained the first nine high-field polynomials in the series for the free energy and for the square and simple cubic lattices the first eleven polynomials.

In § 3 we analyse the high-field magnetisation expansion and low-temperature expansions for the spontaneous magnetisation and initial susceptibility on the square lattice, primarily by Padé approximant techniques. Because the critical point is exactly known we obtain quite precise estimates for the critical exponents and amplitudes. In § 4 we consider evidence concerning the nature of the phase transition. In § 5, assuming the transition to be second order, we obtain estimates for the critical (or tricritical) temperatures, exponents and amplitudes for the simple cubic and body centered cubic lattices.

2. High-field series expansions for the free energy

In this section we describe a method for obtaining a high-field expansion for the three-state Potts model for the special case of the 'non-ordering' field, $H_\perp$ of (1), equal to zero. The method is an adaptation of the shadow lattice method of Sykes *et al* (1965) for the spin-$\frac{1}{2}$ Ising model and applies to bipartite lattices. Introducing the Boltzmann factors $z=\exp(-3\beta J/2)$ and $y=\exp(-3\beta H/2)$ the expansion of the dimensionless free energy per site is of the form

$$f=\sum_n L_n(z)y^n \tag{2}$$

where n is the number of overturned spins and the coefficients $L_n(z)$ are the high-field polynomials.

Enting (1974a) obtained the first five high-field polynomials for the three-state Potts model for non-zero $H_\perp$ on the SC, BCC and FCC lattices. Subsequently Enting (1974b) obtained the first nine and eleven high-field polynomials for $H_\perp=0$ on the square and honeycomb lattices respectively. Enting's work also made use of the shadow lattice method of Sykes *et al* (1965) but in a somewhat different formulation from ours.

To obtain the coefficient of $y^n z^b$ for the spin-$\frac{1}{2}$ Ising model it is necessary to know only the total number of strong embeddings in the lattice of interest of all graphs of n vertices and b edges. For the Potts model the graphs have to be further classified.

Consider a particular graph g formed by turning n spins through an angle $2\pi/3$ radians away from the fully aligned state. The corresponding Boltzmann factor is identical to the Ising Boltzmann factor for the graph, $y^n z^{qn-2b}$, where q is the coordination number of the lattice and $qn-2b$ is the number of 'broken' bonds around the perimeter of the graph. Now turning a subset of the n spins in g through an additional angle of $2\pi/3$ radians does not affect the external or Ising Boltzmann factor of the graph, but does give an additional internal or Potts Boltzmann factor $P(g)$, due to the breaking of the internal bonds. For example the Potts factors for a few of the simplest graphs are: $P(—)=2(1+z)$, $P(\square)=2(1+6z^2+z^4)$ and $P(\square\square)=2(1+6z^2+9z^3+9z^4+6z^5+z^7)$.

The calculation of the Potts Boltzmann factors $P(g)$ for more complicated graphs is greatly simplified by using the first and second Mayer theorems appropriately expressed. If g consists of two disconnected parts g_1 and g_2 it is immediately obvious that

$$P(g)=P(g_1)P(g_2) \tag{3}$$

which is the first Mayer theorem. The second Mayer theorem applies to trees, that is graphs with articulation points.

Consider a graph g formed from two graphs g_1 and g_2, by identifying one of the vertices of g_1 with one of the vertices of g_2, all other edges and vertices remaining distinct. Such a point is an articulation point of the resulting graph, which is a tree

Table 1. Zero-field low-temperature series for the three-state Potts model.

	Simple cubic lattice		BCC lattice	
Degree	Spontaneous magnetisation	Initial susceptibility	Spontaneous magnetisation	Initial susceptibility
0	1	0	1	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	−3	$4\frac{1}{2}$	0	0
7	0	0	0	0
8	0	0	−3	$4\frac{1}{2}$
9	0	0	0	0
10	−18	54	0	0
11	−18	54	0	0
12	42	−126	0	0
13	0	0	0	0
14	−135	$607\frac{1}{2}$	−24	72
15	−270	1215	−24	72
16	477	$-2092\frac{1}{2}$	54	−162
17	648	−2916	0	0
18	−1980	10728	0	0
19	−2988	17928	0	0
20	4140	−23760	−252	1134
21	14052	−83232	−504	2268
22	−21690	145827	900	−4050
23	−52920	367740	1152	−5184
24	55020	−372294	−1452	6750
25	201852	−1482948	−3312	0
26	−162774	1350054	−7344	19872
27	−914538	7376076	11484	44064
28	555750	−4455918	35856	−68634
29	3229524	−27643086	−30132	−215136
30	−1188399	$11261821\frac{1}{2}$	−50184	186462
31	−13301370	124201944	264	308124
32	1402686	−13641102	−113160	51489
33	52334268	−511206966	175464	848700
34			712176	−1305936
35			−319098	−5330304
36			−1997856	2493945
37			334320	15222384
38			341856	−2011284
39			1211472	−12187773
40			13301112	−119221164
41			−2785392	26622792

graph. Now if we fix the orientation of the spin at the articulation point the Potts Boltzmann factor for g_1 is $\frac{1}{2}P(g_1)$, and for g_2 is $\frac{1}{2}P(g_2)$. But this special spin may have either of two perturbed orientations, so the second Mayer theorem becomes

$$P(g)=\tfrac{1}{2}P(g_1)P(g_2). \tag{4}$$

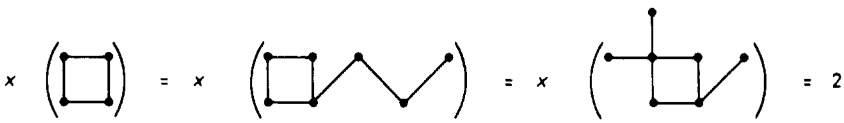
For the r-state Potts model the Mayer theorems remain valid except that the factor $\frac{1}{2}$ in (4) becomes $(r-1)^{-1}$.

For example $P(\square\ )=2(1+z)(1+6z^2+z^4)$. In general if g_2 consists of a Cayley tree (a graph with no closed circuits) of b' bonds then

$$P(g)=2(1+z)^{b'}P(g_1). \tag{5}$$

This equation clearly remains valid if n_2 Cayley trees are attached to g_1 at $n_3 \leqslant n_2$ different articulation points.

Equation (5) has proved very useful in practice and leads to a classification scheme for graphs associated with the Potts model. We define a structure index x, distinguishing all star graphs and star trees, those which contain no Cayley trees as subgraphs. The index, x, for other graphs is then identical to that of the graph obtained by deleting or 'pruning' all Cayley trees. Pure Cayley trees are assigned index $x=1$. Other graphs are ordered first by the number of vertices of g_1 then arbitrarily. For example



Because of (5) it is not necessary to obtain the strong embedding lattice constant of each graph separately, but only the total number of strong embeddings, $[n, b, x]$, for all graphs with n vertices, b edges and structure index x. Then the high-field polynomials are given by

$$L_n(z)=2\sum_{b,x}[n, b, x]P_x(z)(1+z)^{b'_x}z^{qn-2b}. \tag{6}$$

To determine $[n, b, x]$ we adapt the code method of Sykes *et al* (1965) with which we assume familiarity. For illustrative purposes we specialise to the square lattice, although the method applies to any bipartite lattice.

First on the shadow lattice we construct all inequivalent graphs g' through to $n'=5$ vertices for the square and simple cubic lattices and to $n'=4$ vertices for the BCC and honeycomb lattices. The shadow lattice graphs are represented by their adjacency matrices $M(g')$, a record is kept of the number of vertices, the strong embedding lattice constant is found and the code is constructed.

For the example of the shadow lattice graph $/$ on the square lattice $n'(/)=2$, $[/]=2$ and $M(/)=\binom{01}{10}$. There are four vertices on the other sub-lattice, the A sub-lattice, which are neighbours to only one of the above two vertices on B and two on A which are neighbours of both B vertices. Thus the code is (6, 4, 2) exactly as for the Ising model.

The kth shadow lattice graph g'_k has strong lattice constant $[g'_k]$ and code $(\lambda, \alpha, \beta, \ldots)_k$. Hence the partial generating functions of Sykes *et al* (1965) are

$$F_{n'}=\sum_k [g'_k](\lambda, \alpha, \beta, \ldots)_k. \tag{6$'$}$$

Expansion of the shadow lattice codes effectively gives us the required lattice constants on the original lattice, $[n, b, x]$. Incidentally, we do not need to count directly the $[n, b, 1]$ if we make use of the sum rule

$$\sum_{n,b \text{ fixed}} [n, b, x] = [n, b]. \tag{7}$$

Thus through the code method we have

$$[n, b, x] = {}_{-\lambda}C_{n-n'-b+b'} \times {}_{\alpha}C_{b-b'} \times [n', b', x]. \tag{8}$$

Let us return to our $x = 2$ example on the square lattice. There are several graphs on the original lattice with $n = 6$ containing the basic star, the square. However, we have immediately from (8) that the sum of all such strong lattice constants is

$$[6, 5, 2] = {}_{-6}C_1 \times {}_4C_1 \times 1 = -24.$$

The procedure for obtaining the strong embeddings $[n, b, x]$ has been computerised.

The only remaining problem is to obtain the Potts Boltzmann factor for the stars. The number of stars realisable on the lattices studied with eleven or fewer vertices is small enough that the required $P(x)$ could all be computed manually. In practice they were computed by a simple program.

In the Appendix are listed the first nine high-field polynomials for the BCC lattice and the first eleven polynomials for the simple cubic lattice. For each lattice the first five polynomials agree with the results of Enting (1974a). As a further check we have obtained the first nine high-field polynomials on the square and honeycomb lattices, which agree with the results of Enting (1974b). The tenth and eleventh high-field polynomials on the square lattice, also listed in the appendix, are new.

3. Critical behaviour of the Potts model on the square lattice

Although mean field theory and Landau theory predict a first-order transition for the three-state Potts model in two dimensions (Straley and Fisher 1973) it is now known that the model has a second-order transition (Baxter 1973). Thus the reduced spontaneous magnetisation

$$m_0 \approx B_z(1 - z/z_c)^{\beta}. \tag{9}$$

The critical point is also known exactly for the square lattice: $z_c = 0{\cdot}36602\ldots$.

We have not added to the known coefficients in the low-temperature expansion for the spontaneous magnetisation and initial susceptibility of the Potts model on the square lattice. However, we have made a more extensive Padé approximant analysis than previous investigators. (The series are too irregular for ratio analysis.) Knowing z_c exactly, estimates of β can be obtained: (a) from evaluation of Padé approximants to $(z_c - z)(d/dz) \ln m_0$ at $z = z_c$, (b) from a plot of residues versus poles of Padé approximants $(d/dz) \ln m_0(z)$ or (c) from a plot of mean values of poles versus β in Padé approximants to $m_0^{-1/\beta}$.

By method (a) we estimate $\beta = 0{\cdot}106 \pm 0{\cdot}002$. The results of methods (b) and (c) are displayed in figure 2. The position of poles and corresponding residues of high degree, central Padé approximants to $(d/dz) \ln m_0(z)$ are indicated by circles. The lower curve is the best straight line through these points, and yields an estimate of $\beta = 0{\cdot}10652$. The upper curve marks the mean value of poles of Padé approximants to

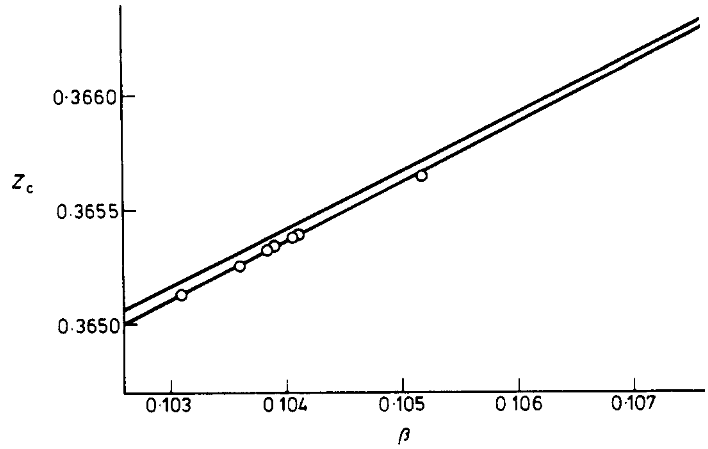


Figure 2. Relation between estimates of z_c and β for the three-state Potts model on the square lattice using high-degree central Padé approximants. Circles locate poles and residues of approximants to $(d/dz)\ln m_0(z)$; the lower curve is the best straight line through these points. The upper curve represents the mean value of poles to $m_0^{-1/\beta}$.

$m_0^{-1/\beta}$. The poles, excluding defective Padé approximants, are closely grouped with a mean deviation of approximately ±0.0001 independent of β in the range illustrated. By this method, (*c*), we estimate $\beta = 0{\cdot}10636$. As our overall best estimate we take

$$\beta = 0{\cdot}1064 \pm 0{\cdot}0005.$$

Previously, by less extensive analysis, Straley and Fisher (1973) estimated $\beta = 0{\cdot}103 \pm 0{\cdot}010$ and Enting (1974b) estimated $\beta = 0{\cdot}105 \pm 0{\cdot}005$. Method (*c*) also yields the amplitude estimate, $B_z = 1{\cdot}200 \pm 0{\cdot}001$.

The initial susceptibility series is less well behaved than the spontaneous magnetisation. Nevertheless, we can estimate γ' and C'_z in

$$\chi_0 \approx C'_z(1 - z/z_c)^{-\gamma'} \tag{10}$$

by the same methods applied to the m_0 series. We find $\gamma' = 1{\cdot}50 \pm 0{\cdot}04$ and $C'_z = 0{\cdot}0050 \pm 0{\cdot}0002$.

Using the estimate of $\alpha = 0{\cdot}296 \pm 0{\cdot}002$ (assumed $= \alpha'$) from Zwanzig and Ramshaw (1977) and our estimate of β and γ', we can test the Rushbrooke inequality. Thus $\alpha' + 2\beta + \gamma' = 2{\cdot}008 \pm 0{\cdot}04$, not only satisfying the inequality but also in agreement with scaling which requires the right side to be 2. Alternatively, assuming the validity of scaling, and using the more precise estimates of α and β we find $\gamma = \gamma' = 1{\cdot}491 \pm 0{\cdot}002$.

Next we analyse the high-field series, where we have eleven terms, two more than Enting (1974b). Again we assume a simple power law singularity of the form

$$m \approx D_y(1 - y)^{1/\delta}. \tag{11}$$

We apply four different techniques to determine δ both in this section and below for the three-dimensional series:

(1) Given the value of z_c residues of Padé approximants to $(d/dy)\ln m(z_c, y)$ are estimates of δ^{-1}.

(2) Knowing that $y_c = 1$ find poles and residues of Padé approximants to $(d/dy) \ln m(z, y)$ for a set of z values. The value of z which yields poles centred at $y = 1$ is then an estimate of z_c and the corresponding residue an estimate of δ^{-1}.

(3) The method used by Gaunt and Sykes (1972) for the Ising model and by Enting (1974a) for the Potts model consists of studying the coefficients in the expansion

$$-y(d/dy) \ln m = \Sigma\, e_n y^n. \tag{12}$$

For $z = z_c$, assumed known from other analysis, $e_n \sim \delta^{-1}$. On a $1/n$ plot the coefficients moreover must approach δ^{-1} with zero slope. Thus by varying z until zero slope is achieved this method also serves as a (rather imprecise) method of estimating z_c.

(4) Assuming z_c to be known, vary δ until poles of Padé approximants to $m(z_c, y)^{-\delta}$ yield $y_c = 1$ as a central estimate.

We have used all four methods on the high-field magnetisation series for the square lattice, and they can also be applied to the high-field susceptibility series, where the exponent δ is replaced by $-\delta/(1-\delta)$. However, for the square lattice we concentrate on the magnetisation series. The final estimates using each of the above methods are listed in table 2.

Table 2. Estimates of the critical properties of the three-state Potts model on the square lattice from analysis of high-field magnetisation series. Underlined numbers are input values.

Method	y_c^Q	z_c^Q	δ
1	$0{\cdot}995 \pm 0{\cdot}005$	$\underline{0{\cdot}36602}$	$16{\cdot}4 \pm 0{\cdot}4$
2	$\underline{1{\cdot}000}$	$0{\cdot}364 \pm 0{\cdot}001$	$17{\cdot}0 \pm 0{\cdot}6$
3	n.a.	$0{\cdot}365 \pm 0{\cdot}001$	$16{\cdot}0 \pm 0{\cdot}5$
4	$\underline{1{\cdot}000}$	$\underline{0{\cdot}36602}$	$15{\cdot}5 \pm 0{\cdot}3$

There is quite a spread among mean estimates by different methods. The 'error bars' in table 2 indicate the scatter of estimates within each method and neglect systematic errors. Note that method 1 underestimates y_c by half a percent while methods 2 and 3 underestimate z_c. Thus we place greatest reliance on method 4 in which the correct values of y_c and z_c are imposed. We adopt the overall estimate of $\delta = 15{\cdot}5 \pm 1{\cdot}5$. From method 4 we also obtain $D_y = 1{\cdot}02 \pm 0{\cdot}01$.

Using our above estimates of β and δ and that of Zwanzig and Ramshaw (1977) for α we find $\alpha + \beta(\delta + 1) = 2{\cdot}05 \pm 0{\cdot}15$ in agreement with the scaling result that the right side is 2. Alternatively using the scaling relation and the estimates for α and β we find $\delta = 15{\cdot}02 \pm 0{\cdot}02$. Thus we are tempted to believe that $\delta = 15$ exactly, as in the two-dimensional Ising model. Enting (1974a) using series of degree nine also concluded that $\delta = 15$.

The low-temperature specific heat series are not sufficiently well behaved to be amenable to the usual methods of analysis.

4. Nature of the transition in three dimensions

We have attempted to determine whether the phase transition is of first or second order. We assume the dimensionless spontaneous magnetisation to behave in the vicinity of the phase transition as

$$m_0 \approx B_0 + B(1 - T/T_c)^{\beta}. \tag{9}$$

Accordingly we have examined the series for $\Delta m = m_0 - B_0$ for a set of values of $0 \cdot 0 \leqslant B_0 \leqslant 0 \cdot 4$. The results of this analysis are summarised in table 3 for the simple cubic lattice. The 'errors' are simply an indication of the degree of convergence among the high degree central Padé approximants. It appears that the convergence is approximately equally good for all values of B_0 in the range examined, so this method cannot be used to determine whether the transition is first or second order. Similar results hold for the BCC lattice.

Table 3. Estimates of z_c and β from poles and residues of Padé approximants to the series for $(d/dz) \ln \Delta m$ where $\Delta m = m_0(z) - B_0$ on the simple cubic lattice.

B_0	z_c	β
0·00	0·5790 ± 0·003	0·21 ± 0·02
0·10	0·5777 ± 0·007	0·22 ± 0·04
0·20	0·5765 ± 0·005	0·26 ± 0·03
0·30	0·5732 ± 0·003	0·33 ± 0·02
0·40	0·5730 ± 0·005	0·33 ± 0·03

Similar Padé approximant studies of the spontaneous magnetisation series on the square lattice, for which it is *known* that the transition of the three-state Potts model is second order (Baxter 1973) are no more definitive than the three-dimensional studies.

It would be still more difficult to estimate the magnitude of a zero-field latent heat. Both high- and low-temperature series for the entropy would have to be used, the latter being quite badly behaved. Moreover, even if there is no latent heat, as in the Ising model, the energy and entropy probably have vertical slopes at the critical point making a continuous transition look very much like a first-order transition.

There remains the possibility of determining the nature of the transition using high-field series. For the square lattice the transition temperature of the Potts model is known exactly (Potts 1952). Accordingly we have studied the series in y of $m(z, y)$ for various values of z. For $z \leqslant 0 \cdot 8 z_c$ evaluation of Padé approximants to the series for $m^{1/\delta}$ ($\delta = 15$) reveals $m^{1/\delta}$ going smoothly to a constant at $y = 1$ as it should. Similarly for $z \geqslant 1 \cdot 5 z_c$ Padé approximants to m go smoothly to zero. In both ranges of z the various Padé approximants yield very consistent values of m for all $0 \leqslant y \leqslant 1$. However in a region $0 \cdot 8\, z_c \leqslant z \leqslant 1 \cdot 5\, z_c$ and $0 \cdot 6 \leqslant y \leqslant 1$, various Padé approximants to the series for either m or $m^{\delta - 1}$ give very scattered estimates for m and little can be said about the location or the nature of the phase transition.

In three dimensions we have an approximate value of z_c from the spontaneous magnetisation series and consequently an approximate value for δ (see below) from the high-field magnetisation series. The same type of analysis of the high-field magnetisation series in three dimensions gives behaviour qualitatively the same and quantitatively neither better nor worse than in two dimensions.

In summary, from Padé approximant analysis of the high-field series and of the low-temperature series we have been unable to determine the nature of the phase transition in the three-state Potts model in three dimensions. We feel that previous attempts by other authors, referred to in the introduction, have likewise failed to determine the nature of the transition. In the remainder of this section, *assuming the transition to be second order*, we determine the critical temperatures, exponents and amplitudes of the Potts model on the simple cubic and BCC lattices.

5. Critical behaviour of the three-dimensional Potts model

To obtain a first approximation to the critical properties of the Potts model in three dimensions we have found poles and residues of high degree, central Padé approximants to $(d/dz)\ln m_0(z)$ and $(d/dz)\ln \chi_0(z)$ on both the simple cubic and BCC lattices. Because of the high degrees of these series, 33 and 41 respectively, the Padé approximants have many poles. In some of these, non-physical poles occur sufficiently near to the physical pole to bias the estimates of critical point and exponent. The decision as to which approximants are sufficiently 'defective' in this way for their poles to be disregarded is somewhat subjective.

Table 4 contains the mean estimates of z_c, β and γ' with 'errors' representing the mean deviation. Badly defective approximants have been excluded. Examination of table 4 reveals that:

(i) estimates of z_c from m_0 are an order of magnitude more precise than those from χ_0; (ii) estimates of the critical exponents β and γ' are barely consistent between the two lattices; (iii) whereas for the simple cubic lattice estimates of z_c from χ_0 are consistent with those from m_0, for the BCC lattice such is hardly the case.

Table 4. Estimates of z_c, β and γ' from poles and residues of Padé approximants to $(d/dz)\ln m_0(z)$ and $(d/dz)\ln \chi_0(z)$ in three dimensions.

Lattice	Function	z_c	β	γ'
simple cubic	m_0	0·5786 ± 0·0003	0·204 ± 0·003	——
	χ_0	0·577 ± 0·002	——	1·10 ± 0·04
BCC	m_0	0·6757 ± 0·0004	0·212 ± 0·003	——
	χ_0	0·672 ± 0·002	——	1·06 ± 0·04

We first concentrate our analysis on the series for the spontaneous magnetisation on the simple cubic lattice, partly because of the results summarised in table 4. In addition, the series on the simple cubic lattice contains information from configurations of up to twelve overturned spins while that for the BCC lattice includes at most nine overturned spins.

We have computed poles and residues of high degree, central Padé approximants to $m_0^{-\beta}$ on the simple cubic lattice for a range of values of β indicated by table 4. The mean value of the poles of non-defective approximants, an estimate of z_c, is plotted as a function of β in figure 3. Poles and residues of $(d/dz)\ln m_0$ are also plotted in figure 3. The error bars associated with the former curve indicate the mean deviation of the poles from the mean value.

The results plotted in figure 3 differ qualitatively from those plotted in figure 2 (for the square lattice) in two important respects. In figure 2 the two trajectories for $z_c(\beta)$ are quite precisely parallel whereas for the simple cubic lattice they intersect. Secondly, for the square lattice the mean deviation of the estimates of z_c from $m_0^{-1/\beta}$ is essentially

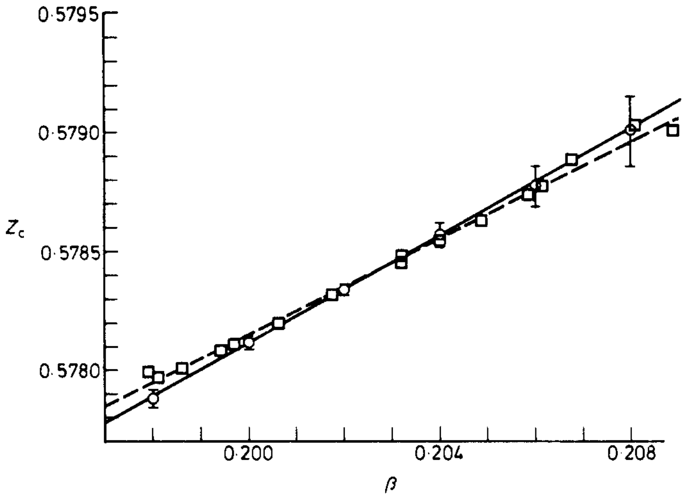


Figure 3. Relation between estimate of z_c and β for the three-state Potts model on the simple cubic lattice using high-degree central Padé approximants. Squares locate poles and residues of approximants to $(d/dz)\ln m_0(z)$; the broken curve is the best straight line through these points. Circles locate mean values of poles to $m_0^{-1/\beta}$ while error bars indicate the mean deviation; the full curve is the best straight line through these points.

independent of β in the range of interest, whereas for the simple cubic lattice the deviation depends strongly on β. The above two features, together with the estimate of β in table 4, allow us to make the estimates of

$$z_c^s = 0{\cdot}5784 \pm 0{\cdot}0003.$$

and

$$\beta = 0{\cdot}203 \pm 0{\cdot}003.$$

Accepting the above value of β the critical point of the BCC lattice can be obtained most precisely from Padé approximants to $m_0^{-1/\beta}$. Corresponding residues also yield estimates of B_z.

The initial susceptibility series have a number q of leading zeros. Thus an estimate of γ' is determined by adjusting γ' in $(\chi_0/z^q)^{1/\gamma'}$ until the mean value of the poles of Padé approximants to the above function coincides with previously obtained estimates of z_c. Residues then yield estimates of C'_z.

From the simple cubic susceptibility we find $\gamma' = 1{\cdot}19 \pm 0{\cdot}01$ while for the BCC lattice we find $\gamma' = 1{\cdot}16 \pm 0{\cdot}02$. Our best estimates of all the above low-temperature critical properties are summarised in table 5.

The only previous series estimates of the critical properties of the three-state Potts model in three dimensions of which we are aware are those of Straley (1974). For the simple cubic lattice only Straley derived degree 24 low-temperature series expansions for the spontaneous magnetisation and initial susceptibility and high-temperature expansions for the zero- field specific heat and initial susceptibility of degrees 10 and 9 respectively. His estimates were $z_c = 0{\cdot}585$, $\beta = 0{\cdot}25 \pm 0{\cdot}05$, $\gamma' = 1{\cdot}3 \pm 0{\cdot}1$ and $\gamma = 0{\cdot}9 \pm 0{\cdot}1$. These less precise estimates, based on less series information, are not inconsistent with our results.

Next we analyse the high-field magnetisation series to determine the exponent δ and amplitude D_y for both lattices. We have used all four methods as described in § 3. In contrast to the situation in two dimensions the exact critical point is not known, so that less weight is to be placed on method 4 results in three dimensions than in two dimensions. Note that both methods 2 and 3 now serve to obtain not only estimates of δ but also additional estimates of z_c.

Table 5. Best estimates of low temperature critical properties of the three-state Potts model in three dimensions.

Lattice	z_c	β	γ'	B_z	C'_z
simple cubic	0·5784 ±0·0004	0·203 ±0·004	1·18 ±0·05	0·958 ±0·003	0·0232 ±0·0006
BCC	0·6747 ±0·0005	0·203†	1·18 ±0·05	0·999 ±0·003	0·0164 ±0·0004

† Value taken from analysis for simple cubic lattice.

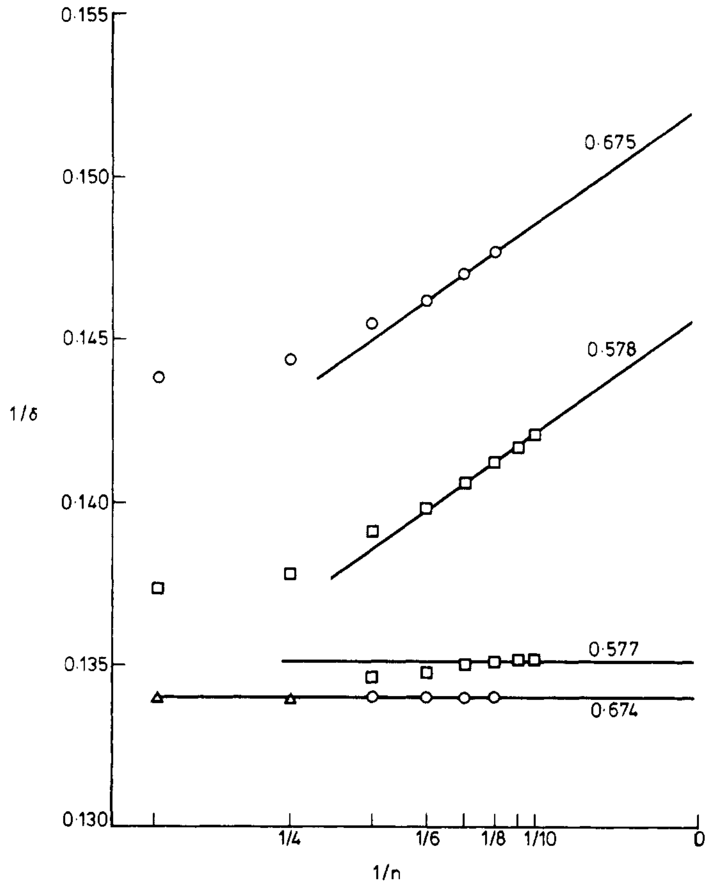


Figure 4. Radio of coefficients of $-y(d/dy) \ln m(z_c, y)$ versus $1/n$ for the simple cubic lattice (squares) and the BCC lattice (circles). Triangles surround points common to both lattices. Input values for z_c are indicated.

Figure 4 is a plot of the coefficients e_n of $-y(\mathrm{d}/\mathrm{d}y)\ln m(y, z_c)$ versus $1/n$, for both the simple cubic and BCC lattices for two different values of z_c for each lattice. For both lattices the estimates of z_c by the criterion that the e_n be asymptotically constant are slightly but distinctly lower than the estimates obtained by analysis of the low-temperature series.

The other three methods of analysis all result in extensive Padé approximant data which we do not reproduce. Instead, the results of all four methods of analysis are summarised in table 6. Error bars represent the mean deviation among estimates from various approximants neglecting any errors in the input (underlined) values. For method 3 error bars represent confidence limits derived from inspection of figure 4. As an overall best estimate we adopt

$$\delta = 7{\cdot}0 \pm 0{\cdot}3.$$

Table 6. Estimates of the critical properties of the three-state Potts model on the three-dimensional lattices from analysis of high-field magnetisation series. Underlined numbers are input values.

Lattice	Method	y_c	z_c	δ
simple cubic	1	0·997 ± 0·005	<u>0·5784</u>	6·8 ± 0·2
	2	<u>1·000</u>	0·5778 ± 0·0005	6·9 ± 0·2
	3	n.a.	0·5770 ± 0·0005	7·4 ± 0·3
	4	<u>1·000</u>	<u>0·5784</u>	6·8 ± 0·2
BCC	1	0·997 ± 0·002	<u>0·6747</u>	7·0 ± 0·1
	2	<u>1·000</u>	0·6740 ± 0·0010	7·1 ± 0·1
	3	n.a.	0·6740 ± 0·0003	7·5 ± 0·3
	4	<u>1·000</u>	<u>0·6747</u>	6·8 ± 0·2

Estimates of D_y for both lattices can be derived from the residues of Padé approximants to $m^{-1/\delta}$ (method 4) in the usual way. We find

$$D_y^S = 1{\cdot}053 \pm 0{\cdot}01$$

and

$$D_y^B = 1{\cdot}017 \pm 0{\cdot}02.$$

We note that the scaling relation $\gamma' = \beta(\delta - 1)$ yields $\gamma' = 1{\cdot}22 \pm 0{\cdot}05$, in agreement with our direct estimates of γ'.

We have also re-analysed Straley's high-temperature series using our presumably more precise value for the critical temperature. The series for the principal susceptibility, χ, was raised to various powers, γ, until the mean value of poles coincided with our estimate of the critical point, $v_c = 0{\cdot}1955$ (instead of Straley's $v_c = 0{\cdot}1977$), was best reproduced. In this way we estimate that $\gamma = 0{\cdot}89 \pm 0{\cdot}05$, in agreement with Straley. As observed by Straley, the scaling relation $\gamma = \gamma'$ appears to be violated. This is rather puzzling as the estimates of β, γ' and δ do satisfy the appropriate scaling relation.

The high-temperature series for the second susceptibility, χ_3 (Straley 1974) is very badly behaved and from Padé approximant analysis we can conclude little about its critical behaviour.

6. Summary and conclusions

The method of high-field expansions has been adapted to be particularly suitable for the three-state Potts model. For the body centred cubic lattice nine complete high-field polynomials in the expansion of the free energy have been obtained, while for the simple cubic lattice eleven polynomials have been obtained. On the square lattice high-field polynomials of degrees ten and eleven have been added. Using the high-field polynomials, low-temperature series for the spontaneous magnetisation and initial susceptibility on the simple cubic and BCC lattices have been obtained for the first time.

For the square lattice, because z_c is known exactly, a very precise estimate of $\beta = 0{\cdot}1064$ has been found, together with somewhat less precise estimates of γ' and δ. Both the direct estimates of δ and the scaling estimate using the above estimate of β and the estimate of Zwanzig and Ramshaw for α strongly suggest that $\delta = 15$, as in the two-dimensional Ising model. If so, then using $\delta = 15$ and the above estimate of β scaling yields $\gamma = \gamma' = 1{\cdot}491 \pm 0{\cdot}002$ and $\alpha = \alpha' = 0{\cdot}296 \pm 0{\cdot}003$; both very close to the direct estimates and well within the error limits.

On three-dimensional lattices the transition temperature is not known exactly, nor is it known whether the transition is first or second order. Assuming the transition to be second order, estimates of β, γ' and δ satisfying with the scaling relation $\beta(\delta - 1) = \gamma'$ can be found. Specifically $\beta = 0{\cdot}203 \pm 0{\cdot}004$, $\delta = 7{\cdot}0 \pm 0{\cdot}3$ and $\gamma' = 1{\cdot}18 \pm 0{\cdot}05$. It is tempting to conjecture that $\delta = 7$ exactly. Certainly the value $\delta = 5$, found exactly for the spherical model and approximately for the Ising, *XY* and Heisenberg models cannot be supported by the series evidence for the three state Potts model.

Acknowledgments

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Appendix

High-field polynomials for the three-state Potts model.

Square lattice

$$L_1 = 2z^4.$$

$$L_2 = 4z^6 - 4z^7 - 10z^8.$$

$$L_3 = 12z^8 + 24z^9 - 52z^{10} - 64z^{11} + 82\tfrac{2}{3}z^{12}.$$

$$L_4 = 2z^8 + 48z^{10} + 108z^{11} - 230z^{12} - 644z^{13} + 604z^{14} + 944z^{15} - 836z^{16}.$$

$$L_5 = 16z^{10} + 16z^{11} + 158z^{12} + 536z^{13} - 1212z^{14} - 4344z^{15} + 2670z^{16} + 13\,216z^{17} - 6544z^{18} - 13\,952z^{19} + 9446\tfrac{2}{5}z^{20}.$$

$$L_6 = 4z^{10} + 104z^{12} + 196z^{13} + 368z^{14} + 2149z^{15} - 5580z^{16} - 27\,988z^{17} + 12\,201\tfrac{1}{3}z^{18} + 115\,420z^{19} - 8644z^{20} - 244\,666\tfrac{2}{3}z^{21} + 62\,336z^{22} + 208\,640z^{23} - 114\,549\tfrac{1}{3}z^{24}.$$

$$L_7 = 44z^{12} + 40z^{13} + 496z^{14} + 1608z^{15} + 44z^{16} + 4224z^{17} - 22\,340z^{18} - 160\,664z^{19}$$
$$+ 38\,000z^{20} + 891\,752z^{21} + 220\,292z^{22} - 2540\,928z^{23} - 618\,744z^{24}$$
$$+ 4290\,816z^{25} - 406\,464z^{26} - 3156\,992z^{27} + 1458\,834\tfrac{2}{7}z^{28}.$$

$$L_8 = 12z^{12} + 352z^{14} + 640z^{15} + 1788z^{16} + 9404z^{17} - 5380z^{18} - 27\,500z^{19} - 91\,332z^{20}$$
$$- 764\,284z^{21} - 22\,312z^{22} + 6161\,852z^{23} + 3553\,124z^{24}$$
$$- 22\,159\,776z^{25} - 15\,513\,044z^{26} + 49\,910\,544z^{27} + 24\,656\,632z^{28}$$
$$- 72\,726\,208z^{29} - 1984\,704z^{30} + 48\,261\,120z^{31} - 19\,260\,960z^{32}.$$

$$L_9 = 2z^{12} + 152z^{14} + 160z^{15} + 1982z^{16} + 6344z^{17} + 4008z^{18} + 35\,976z^{19} - 38\,244z^{20}$$
$$- 407\,776z^{21} - 482\,896z^{22} - 2712\,448z^{23} - 802\,922z^{24}$$
$$+ 37\,177\,304z^{25} + 37\,111\,280z^{26} - 170\,963\,816z^{27} - 201\,574\,722z^{28}$$
$$+ 464\,926\,336z^{29} + 527\,506\,682\tfrac{2}{3}z^{30} - 901\,846\,912z^{31}$$
$$- 664\,857\,312z^{32} + 1202\,720\,938\tfrac{2}{3}z^{33} + 156\,944\,640z^{34}$$
$$- 744\,180\,736z^{35} + 261\,432\,035\tfrac{5}{9}z^{36}.$$

$$L_{10} = 60z^{14} + 24z^{15} + 1272z^{16} + 2728z^{17} + 9064z^{18} + 43\,108z^{19} + 1184z^{20} + 9944z^{21}$$
$$- 231\,592z^{22} - 3083\,340z^{23} - 3372\,678z^{24} - 3616\,328z^{25} + 189\,712z^{26}$$
$$+ 194\,627\,020z^{27} + 295\,499\,788z^{28} - 1160\,333\,048z^{29}$$
$$- 2068\,324\,815\tfrac{1}{5}z^{30} + 3742\,358\,724z^{31} + 7081\,474\,482z^{32}$$
$$- 8421\,032\,448z^{33} - 14\,262\,279\,232z^{34} + 15\,158\,492\,108\tfrac{4}{5}z^{35}$$
$$+ 15\,416\,334\,592z^{36} - 19\,499\,647\,232z^{37} - 4399\,319\,040z^{38}$$
$$+ 11\,559\,174\,144z^{39} - 3626\,978\,304z^{40}.$$

$$L_{11} = 16z^{14} + 716z^{16} + 960z^{17} + 7692z^{18} + 27\,888z^{19} + 35\,500z^{20} + 203\,840z^{21}$$
$$- 57\,524z^{22} - 1226\,008z^{23} - 1894\,240z^{24} - 16\,715\,648z^{25}$$
$$- 23\,407\,104z^{26} + 44\,754\,368z^{27} + 87\,970\,556z^{28} + 864\,715\,232z^{29}$$
$$+ 1841\,229\,740z^{30} - 698\,874\,040z^{31} - 17\,797\,643\,348z^{32}$$
$$+ 26\,003\,925\,536z^{33} + 76\,898\,566\,380z^{34} - 63\,773\,878\,240z^{35}$$
$$- 201\,973\,878\,240z^{36} + 128\,682\,283\,456z^{37} + 341\,191\,214\,976z^{38}$$
$$- 236\,464\,016\,896z^{39} - 329\,153\,510\,784z^{40} + 310\,610\,333\,696^{41}$$
$$+ 99\,404\,160\,000z^{42} - 130\,659\,990\,528z^{43} + 51\,220\,453\,562\tfrac{2}{11}z^{44}.$$

Simple cubic lattice

$L_1 = 2z^6.$

$L_2 = 6z^{10} + 6z^{11} - 14z^{12}.$

$L_3 = 30z^{14} + 60z^{15} - 114z^{16} - 144z^{17} + 170\tfrac{2}{3}z^{18}.$

$$L_4 = 6z^{16} + 202z^{18} + 498z^{19} - 810z^{20} - 2462z^{21} + 1926z^{22} + 3240z^{23} - 2604z^{24}.$$

$$L_5 = 96z^{20} + 96z^{21} + 1308z^{22} + 4464z^{23} - 6728z^{24} - 29\,664z^{25} + 11\,340z^{26}$$
$$+ 77\,296z^{27} - 29\,472z^{28} - 73\,728z^{29} + 45\,008z^{30}.$$

$$L_6 = 36z^{22} + 1208z^{24} + 2292z^{25} + 7698z^{26} + 36\,126z^{27} - 52\,062z^{28} - 340\,680z^{29}$$
$$+ 31\,674z^{30} + 1181\,526z^{31} + 108\,966z^{32} - 2195\,024z^{33}$$
$$+ 352\,032z^{34} + 1707\,840z^{35} - 841\,642\tfrac{2}{3}z^{36}.$$

$$L_7 = 16z^{24} + 804z^{26} + 904z^{27} + 11\,952z^{28} + 36\,600z^{29} + 40\,048z^{30} + 223\,560z^{31}$$
$$- 368\,664z^{32} - 3624\,424z^{33} - 748\,596z^{34} + 15\,976\,368z^{35}$$
$$+ 8035\,880z^{36} - 39\,112\,848z^{37} - 15\,589\,920z^{38} + 59\,171\,776z^{39}$$
$$- 525\,984z^{40} - 40\,207\,104z^{41} + 16\,629\,650\tfrac{2}{7}z^{42}.$$

$$L_8 = 2z^{24} + 16z^{27} + 642z^{28} + 384z^{29} + 11\,832z^{30} + 26\,256z^{31} + 107\,142z^{32}$$
$$+ 433\,212z^{33} + 216\,078z^{34} + 475\,746z^{35} - 2897\,190z^{36}$$
$$- 34\,423\,002z^{37} - 19\,806\,420z^{38} + 196\,907\,622z^{39} + 184\,497\,792z^{40}$$
$$- 581\,180\,322z^{41} - 569\,319\,330z^{42} + 1152\,114\,072z^{43}$$
$$+ 770\,888\,076z^{44} - 1543\,192\,544z^{45} - 172\,358\,496z^{46}$$
$$+ 959\,588\,736z^{47} - 342\,090\,336z^{48}.$$

$$L_9 = 48z^{28} + 48z^{29} + 190z^{30} + 384z^{31} + 13\,848z^{32} + 18\,896z^{33} + 137\,454z^{34}$$
$$+ 470\,088z^{35} + 953\,060z^{36} + 3994\,824z^{37} + 1243\,848z^{38}$$
$$- 12\,959\,680z^{39} - 33\,501\,960z^{40} - 284\,416\,704z^{41} - 287\,126\,802z^{42}$$
$$+ 2191\,512\,648z^{43} + 3208\,207\,692z^{44} - 7656\,444\,104z^{45}$$
$$- 12\,631\,262\,514z^{46} + 17\,479\,072\,368z^{47} + 26\,999\,873\,088z^{48}$$
$$- 30\,932\,324\,352z^{49} - 30\,024\,324\,576z^{50} + 39\,263\,462\,656z^{51}$$
$$+ 8618\,618\,112z^{52} - 23\,163\,091\,968z^{53} + 7257\,873\,464\tfrac{8}{9}z^{54}.$$

$$L_{10} = 48z^{30} + 936z^{32} + 1584z^{33} + 8286z^{34} + 12\,804z^{35} + 216\,404z^{36} + 489\,048z^{37}$$
$$+ 1414\,992z^{38} + 6251\,558z^{39} + 9193\,644z^{40} + 26\,224\,800z^{41}$$
$$+ 886\,276z^{42} - 275\,105\,850z^{43} - 492\,347\,688z^{44} - 1951\,437\,256z^{45}$$
$$- 2845\,414\,176z^{46} + 22\,118\,900\,370z^{47} + 46\,574\,159\,876z^{48}$$
$$- 89\,351\,379\,840z^{49} - 228\,171\,735\,598\tfrac{4}{5}z^{50} + 220\,597\,206\,622z^{51}$$
$$+ 627\,831\,627\,228z^{52} - 429\,195\,997\,728z^{53} - 1077\,708\,256\,000z^{54}$$
$$+ 757\,314\,335\,635\tfrac{1}{5}z^{55} + 1045\,377\,232\,704z^{56} - 977\,956\,798\,720z^{57}$$
$$- 318\,568\,501\,248z^{58} + 564\,451\,547\,136z^{59} - 157\,792\,735\,948\tfrac{4}{5}z^{60}.$$

$$L_{11} = 48z^{32} + 1416z^{34} + 1680z^{35} + 16\,144z^{36} + 37\,968z^{37} + 190\,950z^{38} + 387\,360z^{39}$$
$$+ 2808\,462z^{40} + 8854\,008z^{41} + 14\,633\,802z^{42} + 63\,884\,328z^{43}$$

$+93\,224\,058z^{44}+54\,834\,312z^{45}-246\,325\,302z^{46}-3658\,158\,864z^{47}$
$-6977\,271\,366z^{48}-9325\,587\,384z^{49}-15\,448\,842\,690z^{50}$
$+205\,400\,169\,432z^{51}+579\,280\,677\,702z^{52}-926\,336\,543\,736z^{53}$
$-3565\,427\,509\,344z^{54}+2215\,002\,988\,704z^{55}$
$+11\,983\,233\,128\,664z^{56}-3789\,464\,022\,240z^{57}$
$-26\,312\,202\,402\,528z^{58}+7373\,128\,071\,552z^{59}$
$+3882\,8637\,786\,720z^{60}-16\,477\,939\,592\,448z^{61}$
$-34\,060\,983\,902\,208z^{62}+23\,859\,198\,565\,376z^{63}$
$+104\,900\,408\,40192z^{64}-13\,865\,358\,69\,2352z^{65}$
$-3\,499\,207\,747\,770\tfrac{2}{11}z^{66}.$

Body-centred cubic lattice

$L_1=2z^8.$

$L_2=8z^{14}+8z^{15}-18z^{16}.$

$L_3=56z^{20}+112z^{21}-200z^{22}-256z^{23}+290\tfrac{2}{3}z^{24}.$

$L_4=24z^{24}+552z^{26}+1224z^{27}-1944z^{28}-5976z^{29}+4392z^{30}+7584z^{31}$
$-5860z^{32}.$

$L_5=24z^{28}+504z^{30}+624z^{31}+4588z^{32}+15\,088z^{33}-24\,288z^{34}-95\,936z^{35}$
$+33\,796z^{36}+245\,184z^{37}-86\,208z^{38}-226\,944z^{39}+133\,574\tfrac{2}{5}z^{40}.$

$L_6=54z^{32}+744z^{34}+816z^{35}+7124z^{36}+17\,664z^{37}+32\,648z^{38}+149\,008z^{39}$
$-261\,554z^{40}-1514\,048z^{41}+201\,306\tfrac{2}{3}z^{42}+5013\,888z^{43}+593\,032z^{44}$
$-9116\,309\tfrac{1}{3}z^{45}+1254\,912z^{46}+6913\,152z^{47}-3292\,448z^{48}.$

$L_7=144z^{36}+1744z^{38}+1824z^{39}+12\,948z^{40}+30\,928z^{41}+86\,832z^{42}+288\,528z^{43}$
$+223\,056z^{44}+818\,208z^{45}-2634\,024z^{46}-21\,362\,448z^{47}$
$-2701\,764z^{48}$
$+93\,127\,920z^{49}+44\,689\,320z^{50}-217\,326\,720z^{51}-92\,352\,048z^{52}$
$+321\,965\,824z^{53}+3503\,232z^{54}-214\,061\,568z^{55}+85\,688\,082\tfrac{2}{7}z^{56}.$

$L_8=8z^{38}+396z^{40}+48z^{41}+5104z^{42}+5160z^{43}+31\,324z^{44}+77\,664z^{45}$
$+195\,072z^{46}+627\,560z^{47}+1057\,876z^{48}+3239\,112z^{49}+1065\,304z^{50}$
$-5381\,424z^{51}-33\,867\,324z^{52}-255\,991\,112z^{53}-119\,162\,296z^{54}$
$+1556\,983\,392z^{55}+1352\,897\,168z^{56}-4433\,803\,984z^{57}$
$-4252\,521\,048z^{58}+8363\,498\,240z^{59}+5861\,162\,288z^{60}$
$-11\,001\,254\,784z^{61}-1437\,953\,664z^{62}+6720\,219\,648z^{63}$
$-2321\,129\,760z^{64}.$

$$L_9 = 48z^{42} + 1432z^{44} + 448z^{45} + 14\,864z^{46} + 16\,800z^{47} + 99\,572z^{48} + 230\,288z^{49} + 500\,576z^{50} + 1707\,744z^{51} + 3137\,932z^{52} + 8624\,992z^{53} + 12\,371\,752z^{54} + 21\,283\,504z^{55} - 16\,459\,374z^{56} - 249\,404\,352z^{57} - 547\,954\,168z^{58} - 2593\,488\,112z^{59} - 1983\,374\,101\tfrac{1}{3}z^{60} + 22\,969\,568\,528z^{61} + 31\,440\,400\,416z^{62} - 80\,274\,022\,762\tfrac{2}{3}z^{63} - 125\,603\,876\,666z^{64} + 173\,934\,353\,920z^{65} + 266\,609\,857\,589\tfrac{1}{3}z^{66} - 292\,404\,761\,216z^{67} - 297\,143\,303\,552z^{68} + 366\,590\,492\,330\tfrac{2}{3}z^{69} + 87\,790\,633\,984z^{70} - 213\,402\,370\,048z^{71} + 64\,835\,717\,688\tfrac{8}{9}z^{72}.$$

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